



NAVENVPREDRSCHFAC  
CONTRACTOR REPORT  
*ai* CR 79-07

LIBRARY  
RESEARCH REPORTS DIVISION  
NAVAL POSTGRADUATE SCHOOL  
MONTEREY, CALIFORNIA 93940

# A USE OF TIME SERIES IN IMPROVING WEATHER FORECASTING

Prepared By:

Frank D. Faulkner, Craig Comstock, Robert R. Fossum

U.S. Naval Postgraduate School

*al* Monterey, California 93940

N6658679 WR79064

NOVEMBER 1979

APPROVED FOR PUBLIC RELEASE  
DISTRIBUTION UNLIMITED



Prepared For:

NAVAL ENVIRONMENTAL PREDICTION RESEARCH FACILITY  
MONTEREY, CALIFORNIA 93940

NAVENVPREDRSCHFAC CR 79-07

QUALIFIED REQUESTORS MAY OBTAIN ADDITIONAL COPIES  
FROM THE DEFENSE TECHNICAL INFORMATION CENTER.  
ALL OTHERS SHOULD APPLY TO THE NATIONAL TECHNICAL  
INFORMATION SERVICE.

# ROUTINE REPLY, ENDORSEMENT, TRANSMITTAL OR INFORMATION SHEET

OPNAV 5216/158 (Rev. 1-77)

A WINDOW ENVELOPE MAY BE USED

Formerly NAVI XOS 1989

CLASSIFICATION (UNCLASSIFIED when detached from enclosures, unless otherwise indicated)

UNCLASSIFIED

FROM (Show telephone number in addition to address)

Commanding Officer, Naval Environmental Prediction  
Research Facility, Monterey, CA 93940 AVN 878-2837

DATE

23 January 1980

SUBJECT

NAVENVPREDRSCHFAC technical publication; forwarding of

SERIAL OR FILE NO.

5600 NEPRF/SBB:sb

Ser: 11

REFERENCE

TO:

Distribution  
(on reverse)

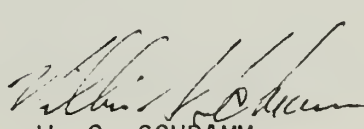
ENCLOSURE

(1) NAVENVPREDRSCHFAC  
Contractor Report  
CR 79-07: A Use Of  
Time Series In  
Improving Weather  
Forecasting

VIA:

ENDORSEMENT ON

☒ FORWARDED ☐ RETURNED ☐ FOLLOW-UP, OR TRACER ☐ REQUEST ☐ SUBMIT ☐ CERTIFY ☐ MAIL ☐ FILE

GENERAL ADMINISTRATION		CONTRACT ADMINISTRATION		PERSONNEL	
FOR APPROPRIATE ACTION		NAME & LOCATION OF SUPPLIER OF SUBJECT ITEMS		REPORTED TO THIS COMMAND:	
UNDER YOUR COGNIZANCE		SUBCONTRACT NO. OF SUBJECT ITEM			
<input checked="" type="checkbox"/> INFORMATION & retention		APPROPRIATION SYMBOL, SUBHEAD, AND CHARGEABLE ACTIVITY		DETACHED FROM THIS COMMAND	
APPROVAL RECOMMENDED <input type="checkbox"/> YES <input type="checkbox"/> NO		SHIPPING AT GOVERNMENT EXPENSE <input type="checkbox"/> YES <input type="checkbox"/> NO		OTHER	
<input type="checkbox"/> APPROVED <input type="checkbox"/> DISAPPROVED		A CERTIFICATE, VICE BILL OF LADING			
COMMENT AND/OR CONCURRENCE		COPIES OF CHANGE ORDERS, AMENDMENT OR MODIFICATION			
CONCUR		CHANGE NOTICE TO SUPPLIER			
LOANED, RETURN BY:		STATUS OF MATERIAL ON PURCHASE DOCUMENT			
SIGN RECEIPT & RETURN					
REPLY TO THE ABOVE BY:					
REFERENCE NOT RECEIVED		REMARKS (Continue on reverse)			
SUBJECT DOCUMENT FORWARDED TO:		<p>Enclosure (1) describes a mathematical technique to identify and correct systematic errors occurring in numerical model weather predictions. A time series of numerical forecasts and their verifying analyses are decomposed into spherical harmonic form. By subtracting the corresponding complex coefficients of the analyses from the forecasts, a time series of forecast error results. If systematic errors in either magnitude or phase occur, a predicted error can be inferred and a subsequent forecast improved by removing that error. Some preliminary results are presented and discussed.</p> <p> W. G. SCHRAMM</p>			
SUBJECT DOCUMENT RETURNED FOR:					
SUBJECT DOCUMENT HAS BEEN REQUESTED, AND WILL BE FORWARDED WHEN RECEIVED					
COPY OF THIS CORRESPONDENCE WITH YOUR REPLY					
ENCLOSURE NOT RECEIVED					
ENCLOSURE FORWARDED AS REQUESTED					
ENCLOSURE RETURNED FOR CORRECTION AS INDICATED					
CORRECTED ENCLOSURE AS REQUESTED					
REMOVE FROM DISTRIBUTION LIST					
REDUCE DISTRIBUTION AMOUNT TO					

COPY TO:

CLASSIFICATION (UNCLASSIFIED when detached from enclosures, unless otherwise indicated)

UNCLASSIFIED

Distribution:

SNDL, Part 1

CINCPACFLT  
COMPHIBGRULANT  
DEPCOMOPTEVFOR (Moffett Field)  
AIRCRAFT CARRIER LANT (USS NIMITZ)  
AIRCRAFT CARRIER PAC (USS ENTERPRISE)  
AMPHIB ASSAULT SHIP LANT (USS IWO JIMA)  
ANTARCTICDEVRO (DET CHRISTCHURCH)

SNDL, Part 2

CNR (Dr. Brodsky)  
ONR (Code 465)  
CNO (OP-952)  
NAVPERS AT DOD AGENCIES (NOAA)  
NAVOCEANCOMDET (Naples, Diego Garcia)  
ONRBRO (Boston, Pasadena)  
NORDA (Dr. Piacsek)  
NAVOCEANCOM  
NAVOCEANO  
FLENUMOCEANCEN  
USNA  
NAVAIRSYSCOM (Codes 954, 370, 553)  
NAVWPNCEN (Dr. Hewer)  
NAVSPASYSACT  
PACMISTESTCEN  
NAVPGSCOL

NAVENVPREDRSCHFAC Master List

- List 3, U.S. Air Force: Items 1, 4, 7.  
List 5, Other DOD Agencies: Items 1, 2, 4, 5.  
List 6, U.S. Coast Guard: Items 2, 4, 7.  
List 7, NOAA: Items 1, 2, 10, 13, 25.  
List 8, Other Government Agencies: Items 1, 2, 4.  
List 9, U.S. Universities: Items 1, 2, 3, 4, 6, 7, 11, 12, 15, 18, 26, 50, 51, and including Univ. of Maryland and Dr. Sasaki.  
List 10, Private Organizations: Items 10, 29.  
List 11, Miscellaneous: Items 1, 2, 5.  
List 12, Foreign Addresses: Items AUS-5; CAN-2; ENG-2, 11; HGKG-1.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NAVENVPREDRSCHFAC Contractor Report CR 79-07	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle)  A Use of Time Series in Improving Weather Forecasting		5. TYPE OF REPORT & PERIOD COVERED  Final
7. AUTHOR(s)  Frank D. Faulkner, Craig Comstock, Robert R. Fossum		6. CONTRACT OR GRANT NUMBER(s)  N6685679WR79064
9. PERFORMING ORGANIZATION NAME AND ADDRESS  U. S. Naval Postgraduate School Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS  PE 62759N, PN WF52551792 NEPRF WU 6.2-6
11. CONTROLLING OFFICE NAME AND ADDRESS  Commander, Naval Air Systems Command Department of the Navy Washington, DC 20361		12. REPORT DATE  November 1979
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)  Naval Environmental Prediction Research Facility Monterey, CA 93940		13. NUMBER OF PAGES  26
		15. SECURITY CLASS. (of this report)  UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)   Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  Time series Regression analysis Weather prediction improvement		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  Given the assumption that weather can be predicted more effectively by integrating the dynamic equation(s) than by examining trends, the question investigated here is whether time series can be used in the secondary routine to effect improvement. The method is described, the relevant equations are derived, a program using real meteorological data is made and run, and a measure of effectiveness is given. Limited study shows some definite promise.		

DD FORM 1473  
1 JAN 73EDITION OF 1 NOV 65 IS OBSOLETE  
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)





## CONTENTS

1.	INTRODUCTION . . . . .	1
2.	OUTLINE OF THE PROBLEM . . . . .	2
3.	INTRODUCTORY TIME SERIES, SIMPLIFYING ASSUMPTIONS . . . . .	3
4.	ROUTINE TO DETERMINE THE BEST FILTERED OR INITIAL VALUES FOR PREDICTION. . . . .	7
5.	TWO SUGGESTED METHODS FOR IMPROVING PREDICTION BY THE USE OF A TIME SERIES . . . . .	10
5.1	Method 1 . . . . .	10
5.2	Method 2 . . . . .	11
6.	COMMENTS . . . . .	13
6.1	Consistency . . . . .	14
6.2	Best Order for the Difference Equation . . . . .	15
6.3	Determining Initial Values . . . . .	16
6.4	Criteria for Adjusting Terms . . . . .	16
6.5	Criteria for Improvement . . . . .	17
6.6	Real Coefficients . . . . .	18
7.	RESULTS . . . . .	18
	SYMBOLS: DEFINITIONS, AND PLACES WHERE INTRODUCED . . . . .	21
	REFERENCES . . . . .	23





## 1. INTRODUCTION

The purpose of this study is to investigate a use of time series in improving weather forecasting.

Let us consider a simple model. Let us assume that the relevant variables, or functions, such as, say, the 500 mb height field, can be expressed in terms of a set of orthogonal basis functions, whose coefficients define the field. There is also an associated numerical integration program which yields predictions of this variable in the form of predicted coefficients. This method of forecasting is assumed to be more reliable than any regression scheme. At regular intervals some observations of the variable are made and the predicted coefficients are then updated, or corrected.

These corrections to the coefficients in the forecast are analyzed for trends. The corrections for each (complex) coefficient of a (complex) eigenfunction or basis function are analyzed separately. The purpose is to predict the corrections.

The basic approach is to define a best-fitting difference equation and a best set of initial conditions, by minimizing weighted sums of squares of the residuals, and then use the difference equation and the initial values to predict the correction. These are used in a secondary routine to improve the forecast.

Testing of the method in the 500 mb field with a simple non-divergent model due to Bourke (1972) shows skill with some coefficients.

## 2. OUTLINE OF THE PROBLEM

Let us assume that we have some function, or functions, which we are representing in terms of an orthonormal set of basis functions. In particular, let us consider the 500 mb height function, or perhaps, the stream function, expressed on the surface of the earth in terms of spherical harmonics. These are expressed as complex trigonometric functions of longitude and Legendre functions of the sine of the latitude. At time  $t_N$  the coefficients of the spherical harmonics form a vector or column matrix  $A_N$ ; its components are complex numbers  $a_{k,N}$ ,  $k=1, \dots, K$ , if the series is truncated to  $K$  terms. The vectors  $A_N$ , and each of the components  $a_{k,N}$ , form a time series, or sequence.

The sequence is generated conceptually as follows. Let us assume that at time  $t_{N-1}$  we have the matrix  $A_{N-1}$  of coefficients for the analyzed field. This is considered to represent the true field at that time. We have also an integration procedure by which we can generate predictions  $A_{N-1,N}$ ,  $A_{N-1,N+1}$ , ..., for the future values  $A_N$ ,  $A_{N+1}$ , ..., as many as we feel are useful or reliable. Then at time  $t_N$ , 12 hours later, observations are made. These are combined with the predictions  $A_{N-1,N}$  to yield a new analyzed field, and a new set of coefficients  $A_N$ . The corrections (changes, errors, or discrepancies),

$$Z_n = A_n - A_{n-1,n}, \quad n = 2, 3, \dots, N, \quad (2.1)$$

also define a time series or a sequence. If we could in some way analyze this sequence,  $Z_2, Z_3, \dots, Z_N$ , and from it predict  $Z_{N+1}, Z_{N+2}, \dots, Z_{N+m}$  reliably then we could predict ahead  $m$  steps

reliably. Our purpose is to predict more accurately, or more reliably, and perhaps to extend the range of prediction. We shall see that there are two ways, at least, that this improvement might be effected. Both depend on the use of time series as developed in the next sections.

The  $Z$ 's and/or their components form the data for the analysis. As such they are called observations (not to be confused with meteorological observations). The terms component and coefficient are used interchangeably. Vectors have components and basis functions have coefficients when they are used in the representation of another function. The components  $a_{k,n}$  of  $A_n$ , for example, are the coefficients in the representation of some meteorological variable. Basis functions are often called modes, particularly in mechanics.

### 3. INTRODUCTORY TIME SERIES, SIMPLIFYING ASSUMPTIONS

Let us consider the sequence of corrections,  $Z_n$ ,  $n = 2, 3, \dots, N$ . The vector  $Z_n$  has  $K$  complex components. There is one complex number  $z_{k,n}$  for each of the  $K$  complex-valued spherical harmonics that is used. To save writing, let  $z_n$ , rather than  $z_{k,n}$ , denote a typical coefficient; we will analyze each of these sequences individually. Part of the rationale is that in a linear uncoupled system the various modes move independently; in a loosely coupled system we would expect similar behavior. There are other important reasons which we will take up later. We shall usually consider  $z_n$  to be a pair of real numbers, a two-by-one matrix.

Let us now assume that  $z_n$  can be expressed as a sum of two terms, described below,

$$z_n = x_n + y_n. \quad (3.1)$$

Here  $y_n$  is a random variable. Its components each are assumed to have expected value zero and expected square  $\sigma^2$ , independent of  $n$ ; they are also assumed to be uncorrelated. The vector  $x_n$  is assumed to be a solution to a difference equation of the form

$$x_n = C_1 x_{n-1} + \dots + C_p x_{n-p}, \quad n = p + 2, \dots, N. \quad (3.2)$$

The coefficients  $C_1, \dots, C_p$  are unknown two-by-two matrices, and the order  $p$  is to be chosen some way. For the present let us take  $p = 2$ , which seems at this time to be a likely value. The  $C$ 's are not constants necessarily, but may vary adaptively with  $N$ .

To determine the coefficients in (3.2), let us choose the  $C$ 's to minimize a residual of the form

$$R = \frac{1}{2} \sum_{n=4}^N w_n \left( z_n - \sum_{j=1}^p C_j z_{n-j} \right)^2, \quad (3.3)$$

where the  $w_n$  are positive weight factors to be chosen or determined; we will discuss this later. To minimize  $R$  let us differentiate with respect to the elements of the  $C$ 's; we get a system of equations of the form

$$\sum_{n=4}^N w_n \begin{pmatrix} z_{n-1} z'_{n-1}, & \dots, & z_{n-1} z'_{n-p} \\ z_{n-p} z'_{n-1}, & \dots, & z_{n-p} z'_{n-p} \end{pmatrix} \begin{pmatrix} C'_1 \\ \dots \\ C'_p \end{pmatrix} = \sum_{n=4}^N w_n \begin{pmatrix} z_{n-1} \\ \dots \\ z_{n-p} \end{pmatrix} z'_n \quad (3.4)$$

the prime (') indicates the transpose of a matrix. The terms like  $z_{n-1} z'_{n-p}$  represent two-by-two matrices so that the elements of the matrices are themselves matrices of order two.

Another problem is the choice of the weight factors  $w_n$ . For a time-independent process we should use equal weightings, which simplifies several relations. However the fundamental relations may vary rather rapidly with time sometimes, and these are the very times that are most critical. One way to take account of this is to use exponential weightings in (3.3) and (3.4). After some starting routine we weight each new observation with a uniform value  $w$  and decrease all earlier weightings then by a factor  $1-w$ . We do this as follows. Let  $Q_N$  be the matrix of coefficients in (3.4); it has two-by-two block elements of the form

$$Q_{i,j,N} = \sum_{p+2}^N w_n z_{n-i} z'_{n-j}, \quad i,j=1, \dots, p; \quad (3.5)$$

the right-hand side of (3.4) has elements  $Q_{i,0,N}$ . When we get the observation  $z_N$ , we update  $Q_{i,j,N-1}$  by the relation

$$Q_{i,j,N} = Q_{i,j,N-1} + w(z_{N-i} z'_{N-j} - Q_{i,j,N-1}). \quad (3.6)$$

We may actually update  $Q$  by first shifting each block element down and to the right one place, dropping out the last row and the last column. Then we update the first (block) row, using (3.6), and then the first column by symmetry. The weight of an observation from time  $t_n$  for the coefficients calculated at time  $t_N$  is  $w(1-w)^{N-n}$ ; we see that weightings decrease exponentially with time.



The best choice of  $w$  is another problem. If  $w$  is too small the system is slow to respond to changes, and if  $w$  is too large, random errors cause excessive errors in the predicted values.

The above procedure for updating the coefficients in (3.2) is rather efficient, both in terms of storage and computation. The matrix  $Q$  is always singular until at least  $2p$  observations have been made. It is usually positive definite after that, but it was sometimes found to be near singular, especially for the smaller values of  $N$ .

In order to predict the  $z$ 's ahead, using (3.2), we also need  $p$  starting or initial values; we can take these conceptionally as estimates of  $x_N, x_{n-1}, \dots, x_{N-p+1}$ . The determination of these is taken up in the next section; it requires more storage and more computation than the above.

In the following section we will generally consider the case  $p = 2$ .

There may be some minor discrepancies in the first index; for example, when the  $a$ 's are involved we tend to think of  $z_2$  as the first of the  $z$ 's, and later, when treating the  $z$ 's we think of the first as  $z_1$ .



#### 4. ROUTINE TO DETERMINE THE BEST FILTERED OR INITIAL VALUES FOR PREDICTION

Let us assume that we have found the coefficients  $C_j$  in the difference Eq. (3.2). We also need estimates of  $x_N$  and  $x_{N-1}$ , say  $\hat{x}_N$ ,  $\hat{x}_{N-1}$ , in order to predict ahead using the difference equation. We might just use the last two observations,  $z_N$ ,  $z_{N-1}$ , but these involve individual random errors, and we expect smoothed or filtered values to be more reliable. By using a large value for the weighting parameter  $w$  we can drive the smoothed values close to the last two observations, if we wish.

Let us resolve the problem as follows. Let us consider

$$\begin{aligned} y_n &= z_n - x_n & (n = 1, \dots, N) \\ &= z_n - (C_1 x_{n-1} + C_2 x_{n-2}) & (n = 3, \dots, N) \\ &= z_n - (C_1 [C_1 x_{n-2} + C_2 x_{n-3}] + C_2 [C_1 x_{n-3} + C_2 x_{n-4}]) \\ &= \dots, \end{aligned}$$

which we may rewrite in the form

$$y_n = z_n - A_n x_2 - B_n x_1, \quad (n = 1, \dots, N); \quad (4.1)$$

$A_n$  and  $B_n$  are defined by this relation. It is easily verified that they satisfy the relations

$$\left\{ \begin{aligned} A_n &= C_1 A_{n-1} + C_2 A_{n-2} & n = 1, 2, \dots, N, \\ A_1 &= 0, A_2 = I, \\ B_n &= C_1 B_{n-1} + C_2 B_{n-2} \\ B_1 &= I, B_2 = 0. \end{aligned} \right. \quad (4.2)$$

We can simplify the programming somewhat if we define the matrices

$$\begin{cases} D^n = (A_n, B_n) \\ U = (x_2', x_1')' = (x_{12}, x_{22}, x_{11}, x_{21})'; \end{cases} \quad (4.3)$$

$x_{ij}$  is the  $i$ 'th component of  $x_j$ . Then  $D^n$  satisfies Eq. (3.2) and has the initial values

$$\begin{cases} D^n = C_1 D^{n-1} + C_2 D^{n-2} \\ D^1 = (0, I), D^2 = (I, 0). \end{cases} \quad (4.4)$$

Equation (4.1) now reduces to

$$y_n = z_n - D^n U. \quad (4.5)$$

If we knew the four components of  $U$ , we could get starting values  $\hat{x}_N, \hat{x}_{N-1}$  from Eq. (3.2).

$$x_n = C_1 x_{n-1} + C_2 x_{n-2}, \quad (4.6)$$

and if we select or estimate  $U$  in some way we get corresponding estimates for  $x_N, x_{N-1}$ . Of course it would be nicer to determine these in a more direct way, but there seems to be no way to do this.

Now, by assumption, the expected value,  $E y_i = 0$ . Hence it seems reasonable that the best choice for  $U$  is one which minimizes a weighted sum of squares for the estimates of  $y_i$ , say,

$$R = \frac{1}{2} \sum_{n=1}^N w_n y_n^2 = \frac{1}{2} \sum_{n=1}^N w_n (D^n U - z_n)^2 \quad (4.7)$$

To effect this we differentiate with respect to the components of  $U$ ; if we call its components  $u_j$ , the  $j$ 'th partial derivative yields

$$\begin{aligned} R_j &= \sum_{n=1}^N w_n \left[ d_{1,j}^n \left( \sum_{k=1}^4 d_{1,k}^n u_k - z_{1,n} \right) + d_{2,j}^n \left( \sum_{k=1}^4 d_{2,k}^n u_k - z_{2,n} \right) \right] \\ &= 0, \end{aligned} \quad (4.8)$$

where  $d_{j,k}^n$  are the components  $D^n$ . We solve these equations for  $U$ , which defines  $\hat{x}_1, \hat{x}_2$ , and from these we get our best estimates,  $\hat{x}_{N-1}, \hat{x}_N$  for starting.

There are several problems. First, the sequence  $z_n$ ,  $n = 1, 2, \dots, N$ , may be very long, so that computing times and errors may be significant. Further the observations  $z_n$  for small values of  $n$  may not be relevant to today's weather. We also need to recall many observations. These problems suggest limiting the range of summation in (4.7) to, say, the last ten terms, which we did.

Another problem needs some explanation. We really just need  $\hat{x}_{N-1}, \hat{x}_N$  to predict ahead. However we cannot get these directly. It really does not matter whether we find  $\hat{x}_{N-1}$  and  $\hat{x}_N$ , or  $\hat{x}_{N-9}$  and  $\hat{x}_{N-8}$ ; we get the former from the latter by use of difference Eq. (3.2). We might perhaps try to rewrite the difference equation, to solve for  $x_{n-2}$  in terms of  $x_{n-1}$  and  $x_n$ , to avoid this. However we must invert the matrix  $C_2$  to effect this, and there is no reason that it cannot be singular, or nearly so, randomly.

In the program we have 15  $\hat{x}$ 's actually indexed from 1 to 15. The first 10 of these are the 10 approximants to the last 10 observations  $z_{N-9}, \dots, z_N$ ; the last five are predictions for the future values  $z_{N+1}, \dots, z_{N+5}$ .

## 5. TWO SUGGESTED METHODS FOR IMPROVING PREDICTION BY THE USE OF A TIME SERIES

There are two particular methods which seem feasible for improving the forecast. Each of these seems feasible, but each poses some problems.

### 5.1 Method 1

Let us consider again the matrices for the coefficients in the analyzed fields,  $A_n$ ,  $n=1, \dots, N$ ; we wish to improve the predictions of these. Let us assume that we have generated and stored the following. First, we have stored the last ten corrections  $Z_{N-9}, \dots, Z_N$ . We have also stored, for each term in the  $Z$ 's, the matrix  $Q$  which defines the coefficients  $C_j$  in the difference equation.

The routine is the following for each component of  $Z$ . As soon as the new value of  $Z$ ,  $Z_N$ , is obtained, we store it and discard the old  $Z_{N-9}$ . Then we update the matrix  $Q$  and solve for the coefficients of the difference equation  $C_j$  as described in Section 3. Then we solve for the initial values  $\hat{x}_{N-1}, \hat{x}_N$ , as described in Sect. 4, and finally we predict ahead to get  $x_{N+1}, x_{N+2}, \dots, \hat{x}_{N+M}$ , as many as we wish. These define the  $\hat{x}$ 's, or  $\hat{X}$ 's.

Next let us use the dynamic equation and integrate ahead one time step, to get an estimate  $A_{N,N+1}$ , of  $A_{N+1}$ . Then let, say,

$$\tilde{A}_{N,N+1} = A_{N,N+1},$$

and

$$\hat{A}_{N,N+1} = \tilde{A}_{N,N+1} + \hat{X}_{N+1}. \quad (5.1)$$

We get successive values in a similar way. When we have  $\hat{A}_{N,N+j}$  we integrate to obtain  $\tilde{A}_{N,N+j+1}$ , a first estimate of  $A_{N,N+j+1}$ . Then we add  $\hat{X}_{N+j+1}$ , to get

$$\hat{A}_{N,N+j+1} = \tilde{A}_{N,N+j+1} + \hat{X}_{N+j+1}. \quad (5.2)$$

With the initial value  $\hat{A}_{N,N} = A_N$ , the sequence of predictions, or forecasts, is defined.

In the initial part of the study, a simple integration routine was used for predicting,  $A_{N,N+M}$ , a model using the non-divergent barotropic vorticity equation. When this model is used, the method of correcting above, alternately adjusting and integrating in feasible and practical.

When a more accurate and complicated weather prediction model is used, serious difficulties arise with this method. Restarting a time integration after rather arbitrary adjustments to one or more variables may generate spurious gravity waves which degrade the prediction and offset the adjustments. To eliminate this difficulty a second method may be used, in which a separate series is generated for each interval of prediction. The appropriate term is used for each interval and the adjusted functions are never integrated.

## 5.2 Method 2

There is another method which gets around the above problem, at the expense of increasing the storage and computation by a factor of roughly  $M$ , if we use  $M$  different intervals of prediction. That is, if we wish, say to adjust the 12-, 24-, and 36-hour forecasts we must generate three time series.



Now assume that we wish to predict ahead  $M$  steps. At the time  $t_N$  we will have the predictions based on integration,

$$A_{N,N+1}, \dots, A_{N,N+M} = \{A_{N,N+m}\}_{m=1,M} \quad (5.3)$$

For the corrections we will save  $M$  sequences  $Z_n^m$

$$Z_n^m = A_n - A_{n-m,n}, \quad m = 1, \dots, M, \quad n = m+1, \dots, N. \quad (5.4)$$

Let us consider any particular value of  $m$ . To generate the data for the time series we must store each time the uncorrected predictions  $A_{n,n+m}$ . Then at time  $t_{n+m}$  we will obtain and store  $Z_{n+m}^m$  from (5.4).

We will thus have  $M$  time series to be analyzed. Each is analyzed as discussed earlier. In this case however we will use a single prediction from each one: from  $Z_n^m$  we generate the single correction for  $m$  time steps ahead. If we denote the corrections by  $\hat{X}_n^m$  the modified values will be

$$\hat{A}_{n,n+m} = A_{n,n+m} + \hat{X}_n^m \quad (5.5)$$

This method has the advantage that the modified terms are not integrated. It has the disadvantage that it requires one time series for each value of  $m$  used, which leads to larger storage and computational requirements.



## 6. COMMENTS

The problem here differs from the most common applications of time series. We have a rather small amount of information which tends to be masked by a large random element. From this we are trying to predict transients, or trends, for relatively short periods, perhaps one to ten time steps. We are not directly interested in the smoothed values, since we are not concerned with what has happened, except as our ability to fit it reflects on our ability to predict. Our feeling is that the variables we are predicting may change rather rapidly so that the weather two weeks ago is of little interest; even the equation which governs the behavior may well have changed. We cannot really make use of long-term observations, as for a stable system, which allow more accurate predictions.

We are trying to predict the short-range behavior of a part of a non-linear system of high order by solutions to a linear system of low order. Whether we try to do this by a difference equation or by Taylor series depends on the type of data, and the nature of the expected solutions. In our case both the form of the data and the anticipated periodic properties of the expected solutions suggest difference equations.

The papers by Jones (1963), (1964) particularly, and (1965) suggested the general problem here. These treat more the descriptive and the long-range forecasting problem. The series are considered to be stationary in the sense that all observations are weighted equally; several formulas are then simplified.

The references found most useful in this study were Whittle (1963) and Gelb (1974). Recent developments in the use of spherical harmonics in meteorology (see GARP, 1974) and Fast Fourier Transforms make the methods appear feasible.

The method is clearly that of least squares. It is also a linear regression method; all of the equations to be solved are linear. If we were to use, say, our estimates  $\hat{x}_{N-1}$ ,  $\hat{x}_N$  in a routine to try to improve the C's, then it would be non linear. The solutions to the difference equations are basically complex exponentials, that is, a combination of real exponentials and trigonometric functions.

There are a number of points that should be discussed or clarified.

#### 6.1 Consistency

It should be pointed out that even for a stationary system the solutions are not consistent, as follows. Let us assume that we have a solution to a difference Eq. (3.1) to which random uncorrelated terms  $y_n$ , each with expected value 0 and expected square  $\sigma^2$  is added. Then Eq. (3.4) will not yield the desired coefficients in (3.2). The expected elements of the matrix of coefficients  $Q$  in (3.5), from (3.4) will be

$$Q_{i,j,N} = \sum_n^N w_n (x_{n-i} x'_{n-j} + \delta_{ij} \sigma^2 I_2) \quad i,j=1, \dots, p; \quad (6.1)$$

here  $I_2$  is the second-order identity matrix, and  $\delta$  is the Kronecker  $\delta$ , (=1 when  $i=j$  and 0 when  $i \neq j$ ). That is, the  $Q$  matrices have an extra term  $\sigma^2$  on the main diagonal. This suggests that if we have a measure of  $\sigma^2$  we could subtract it from the elements on the main

diagonal of  $Q$ , in a sort of negative ridge-regression scheme. This is very risky procedure however, since it drives the matrix of coefficients toward singularity. Difficulty has been encountered several times because this matrix  $Q$  was singular, and the danger is aggravated when exponential weighting with a large decay rate is used.

## 6.2 Best Order for the Difference Equation

We have suggested that we use a second-order difference equation,  $p = 2$ . Some of the reasons follow. Let us consider a system which is time independent, and assume that we have found the correct order  $p$  and the true coefficients  $C$  in the difference equations. In this case the residuals yield an estimate for  $\sigma^2$ . Now consider the residual

$$R = \sum_{n=p+1}^N w_n \{x_n + y_n - \sum_{j=1}^p C_j (x_{n-j} + y_{n-j})\}^2$$

$$= \sum w_n (y_n - \sum C_j y_{n-j})^2,$$

since the  $x$ 's satisfy (3.2). The expected value of the residual is then

$$E(R) = (\sum w_n) (2 + \sum c_{j,k,i}^2) \sigma^2,$$

where the last term in the parenthesis denotes the sum of the squares of all of the components of all of the  $C$ 's.

Now if we try to fit the solutions to a second-order system by a third-order difference equation, in the absence of noise, the matrix of coefficients is singular, since the solutions satisfy many third-order difference equations. If we have a little noise

we may have a near-singular matrix and have large values for the  $c_{i,j,k}$ ; in this case we get a large expected value in (6.1) for the residual. Limited numerical checks make the second-order equation appear satisfactory.

### 6.3 Determining Initial Values

There was no particular reason for choosing ten values to fit when we determine the starting values for prediction. It was felt that six would be the smallest number to be considered, that fewer would make the methods vulnerable to random errors in the last terms. Many more than ten could cause roundoff errors.

### 6.4 Criteria for Adjusting Terms

We do not expect to adjust or correct all of the coefficients. For some the errors may be too small to warrant the time and trouble. Other coefficients may contain such random elements and change so unpredictably that trends are effectively obscured. The decision to adjust or correct will probably depend on an analysis of similar sequences in the past. There are two basic ways to do this, one based on a long sequence of past data (that is, on climatology), and the other on recent data (corresponding to recent weather).

We may analyze a long sequence for a coefficient, perhaps over a period of years, and from this make a decision, beforehand always to correct, or never, in the program.

An alternate way is to see how well the smoothed values fit the last ten corrections. We generate these values naturally in determining the starting values for prediction, so this entails little computation. We may consider, for example, the ratio



$$1.25 \frac{\sum_1^{10} w_n (z_{N-10+n} - \hat{x}_n)^2}{\sum_1^{10} w_n z_{N-10+n}^2} .$$

If this is much less than one we have fitted the past ten values well, and if it exceeds one we have no skill in fitting the observations. The factor 1.25 adjusts for the fact that we could fit two terms exactly with the initial conditions. We might correct or not accordingly as this number was below or above some criterion, say, .75. The philosophy is that if we can fit a sequence of observations well, then we can predict well. (More logical is the converse; if we cannot fit the observed values, then we cannot predict well.) It is probably not worthwhile to use this criterion unless we usually correct.

#### 6.5 Criteria for Improvement

In the numerical study we used two principal criteria for improvement. Both of these compared in some way the magnitude of the errors in prediction with adjustment with the errors generated without the adjustment. In the first criterion, the error in the predicted values(s) was compared with the rms value of the last ten corrections (without adjustment). This figure was calculated as the predictions were under way. For the second criterion, a long sequence of adjustments were calculated. Then the rms values of the errors in prediction after adjustment was compared with the rms value of the error in prediction without adjustment:

$$\sum |z_n - \hat{x}_n|^2 / \sum |z_n|^2;$$

the range of summation of n is over each term for which a correction to z was made. These measures are easily made in the machine, and they define a simple norm for the error relation.

It is not clear these norms are good ones from a meteorological point of view. Very likely the ultimate test will rest on maps and criteria for judging them.

#### 6.6 Real Coefficients

Most of the coefficients are complex. The one associated with the modes that are independent of longitude (meridional) are real and hence one dimensional. The routines developed then involve scalar equations, rather than two-dimensional vectors. The corresponding spherical harmonics are the Legendre polynomials, the simplest of the Legendre functions.

### 7. RESULTS

The procedure for analyzing the time series was checked on several sets of made-up data, and on two coefficients  $a_2$  and  $a_4$  (of  $P_1^1$  and  $P_2^2$ ), using a difference equation of second order,  $p = 2$ . For data made up of solutions to a difference equation, with random noise added, the solution could be recovered if the noise was not too large, but the routine did not perform very well when the energy in the noise (rms value) was larger than in the solution.

For meteorological data, on a string of 60 pairs of corrections, the predictions  $x_{11}$  to  $x_{55}$  led to a definite improvement for one coefficient. The best value for the weighting  $w$  of the new values seemed rather small: the values .025, .05 and .1 gave similar results and were better than larger values. The numbers below were determined as follows. We considered the coefficient of  $P_1^1$ . A set of 10 data points were used at the start to get coefficients for the best difference equation and the initial



values. Five values were predicted, corresponding to the 12-hour correction to be made in 12 hours, 24 hours, ..., 60 hours. The error in predicting these was then calculated and normalized by dividing by the rms of the 10 original data. Then the time index was advanced by one and the operation repeated.

The numbers in row one are the rms values of  $a_{n+1} - a_{n,n+1}$ , normalized by the rms values of  $z_{n-9}, z_{n-8}, \dots, z_n$ . The numbers in row m denote the rms value of  $a_{n+m} - a_{n+m-1,n+m}$ , normalized by the rms value of  $z_{n-9}, \dots, z_n$ . Each is then a correction to be applied after integrating one time step. Each entry comes from the sum of 46 terms.

No. of Time Steps	Weight w	.025	.05	.1	.2	.4
1		.6239	.6257	.6131	.6404	.8138
2		.6370	.6401	.6312	.6629	.8640
3		.6670	.6751	.6834	.7599	1.3095
4		.6573	.6698	.6960	.8314	1.5523
5		.6513	.6636	.6880	.8496	1.9109

There are several interesting points about the data. First, the results from the small values of w, from .025 to .1 are all comparable. This indicates a stable trend: when  $w = .025$  the first of the 10 data points had a weighting almost .8 of the last; they are weighted nearly equally. For  $w = .10$  the relative weighting is about .39. A surprising feature is that all five of the predictions are so similar. This is further indication of a stable trend, a large "random" element, and a satisfactory value of p, and suggests longer predictions are feasible.

The corrections to the fourth coefficient were not predicted successfully; the error in the predicted values was consistently a little large than the corrections. However various features of the solutions are quite similar to the other solutions; the smaller values of  $w$  give better predictions and all five predicted values are similar in size, with later predictions often anomalously having smaller error than the first.

#### Acknowledgments

We are particularly indebted to Dr. C. E. Leith of NCAR, Boulder, for making available to us both a large set of data from analyzed fields and an integration program for generating data to try this method. We are also greatly indebted to our colleague, Professor H. J. Larson, for a number of clarifying discussions.

## SYMBOLS: DEFINITIONS, AND PLACES WHERE INTRODUCED

### Section 2

$t_n, t_N$	time $N$ is usually associated with the last or next-to-last observation.
$A_n$	the matrix, or vector, of coefficients for the analyzed field at time $t_n$ .
$a_{k,n}$	the $k$ 'th element of $A_n$ , $k=1, \dots, K$ . Most of the $a_{k,n}$ are complex numbers and are treated like two-dimensional vectors.
$K$	the number of spherical harmonics used to represent a function.
$a_n (=a_{k,n})$	a typical element of $A_n$ , written without index $k$ to save writing.
$A_{n,n_1}$	the value predicted for $A_{n_1}$ at time $t_n$ .
$Z_n = A_n - A_{n-1,n}$	This represents the discrepancy between the analyzed field at $t_n$ and the value predicted for it at time $t_{n-1}$ . Also called a correction.

### Section 3

$z_n$	a typical element of $Z_n$ .
$y_n$	the component of $z_n$ that is assumed to be random.
$x_n$	the component of $z_n$ that is assumed to satisfy a difference equation (see below).
$C_j$	two by two matrices in the difference equation $x_n = C_1 x_{n-1} + \dots + C_p x_{n-p}$
$R$	weighted sum of residual errors.
$w$	weighting of new terms.
$w_n$	the weighting of term associated with $t_n$ , at time $N \geq n$ ; $w_n$ decreases with $N$ for a fixed value of $n$ : $w_n = w(1-w)^{N-n}$ .

### Section 3 (continued)

$\hat{x}_n$  a best approximation for  $x_n$ , made at some definite time  $t_N$ . (It might also have been designated by  $\hat{x}_{k,N,n}$ .)

$\hat{X}_n$  the vector, or matrix, of  $\hat{x}_n$ 's.

### Section 4

$A_n, B_n$  matrices defined in Eq. (4.2), used in finding  $x_n$ 's.

$D^n = (A_n, B_n)$  (see Eqs. (4.3), (4.4)).

$x_{ij}$  the  $i$ 'th component of  $x_j$ ,  $i, j=1, 2$ .

$U = (x_{12}, x_{22}, x_{11}, x_{12})'$

### Section 5

$\hat{A}_{N,N+m}$  an optimal predictor for  $A_{N+m}$ , made at  $t_N$ .

$\tilde{A}_{N,N+m}$  a temporary estimate of  $A_{N,N+m}$ , used in obtaining  $\hat{A}_{N,N+m}$ .

$Z_n^m = A_n - A_{n-m,n}$

$z_n^m$  typical elements of  $Z_n^m$ .

$R$  a general symbol for a quadratic residual to be minimized.

## REFERENCES

- Bourke, William, 1973: An efficient, one-level, primitive-equation, spectral model, Mon. Wea. Rev., 100, 683-689.
- GARP/WGNE, 1974: Global Atmospheric Research Program/Working Group on Numerical Experimentation, Report of International Symposium on Spectral Methods in Numerical Weather Prediction, 12-16 August 1974, Copenhagen, Rept. No. 7.
- Gelb, Arthur, editor, 1974: Applied Optimal Estimation, Cambridge, MA, MIT Press.
- Jones, Richard H., 1962: Stochastic Processes on a Sphere as Applied to Meteorological 500-Millibar Forecasts, Ch. 7, (Proceedings of the Symposium on) Time Series Analysis, Murray Rosenblatt, John Wiley, NY, 1862.
- \_\_\_\_\_, 1964: Prediction of multivariate time series, J. Appl. Meteor., 3, 285-289.
- \_\_\_\_\_, 1965: Optimal estimation of initial condition for numerical prediction, J. Atmos. Sci., 22, 658-663.
- Whittle, Peter, 1963: Prediction and Regulation, Princeton, D Van Nostrand.





U190183

DUDLEY KNOX LIBRARY - RESEARCH REPORTS



5 6853 01069347 6

U190183